PROPER HOLOMORPHIC MAPPINGS BETWEEN COMPLEX ELLIPSOIDS AND GENERALIZED HARTOGS TRIANGLES

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Abstract. The explicit form of proper holomorphic mappings between complex ellipsoids is given. Using this description, we characterize the existence of proper holomorphic mappings between generalized Hartogs triangles and give their explicit form. In particular, the automorphism group of such domains is found.

For any bounded domains $D, G \subset \mathbb{C}^n$ let $\operatorname{Prop}(D, G)$ denote the set of proper holomorphic mappings $F : D \to G$, where proper, as usual, means $F^{-1}(K)$ compact in D for every compact $K \subset G$. In this context, the compactness of $F^{-1}(K)$ for every compact $K \subset G$ is equivalent to the following condition: for any sequence $(z_\nu)_{\nu\in\mathbb{N}} \subset D$ that has no limit point in D the sequence $(F(z_\nu))_{\nu\in\mathbb{N}} \subset G$ has no limit point in G. Let $\operatorname{Aut}(D)$ denote the automorphism group of D, i.e. the set of all biholomorphic self-mappings $F : D \to D$. Moreover, we shall write $\operatorname{Prop}(D) := \operatorname{Prop}(D, D)$.

Our aim is to characterize the sets $\operatorname{Prop}(D, G)$ and $\operatorname{Aut}(D)$ when D, G belong either to the class of the complex ellipsoids or the so-called generalized Hartogs triangles.

Here is some notation. Let Σ_n denote the group of the permutations of the set $\{1, \dots, n\}$. For $\sigma \in \Sigma_n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ denote $z_\sigma := (z_{\sigma(1)}, \dots, z_{\sigma(n)})$ and $\Sigma_n(z) := \{\sigma \in \Sigma_n : z_\sigma = z\}$. We shall also write $\sigma(z) := z_\sigma$.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{>0}$ put

$$\Psi_\alpha(z) := z^\alpha := (z_1^{\alpha_1}, \dots, z_n^{\alpha_n}), \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

If, moreover, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n_{>0}$ we shall write $\alpha\beta := (\alpha_1\beta_1, \dots, \alpha_n\beta_n)$ and $1/\beta := (1/\beta_1, \dots, 1/\beta_n)$.

For $z \in \mathbb{C}$, $A \subset \mathbb{C}$ let $zA := \{za : a \in A\}$.

Finally, let $\mathbb{U}(n)$ denote the set of unitary mappings $U : \mathbb{C}^n \to \mathbb{C}^n$.

1. Complex ellipsoids

For $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$, $n \geq 2$, define the *complex ellipsoid*

$$\mathbb{E}_p := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1 \right\}.$$

Note that $\mathbb{B}_n := \mathbb{E}_{(1,\dots,1)}$ is the unit ball in $\mathbb{C}^n$. We shall write $\mathbb{D} := \mathbb{B}_1$, $\mathbb{T} := \partial\mathbb{D}$. Moreover, if $\alpha/\beta \in \mathbb{N}^n$ then $\Psi_{\alpha/\beta} \in \operatorname{Prop}(\mathbb{E}_\alpha, \mathbb{E}_\beta)$.

The problem of characterization of $\operatorname{Prop}(\mathbb{E}_p, \mathbb{E}_q)$ and $\operatorname{Aut}(\mathbb{E}_p)$ has been investigated in [9] and [6]. The question on non-emptiness of $\operatorname{Prop}(\mathbb{E}_p, \mathbb{E}_q)$ as well as the form of $\operatorname{Prop}(\mathbb{E}_p, \mathbb{E}_q)$ and $\operatorname{Aut}(\mathbb{E}_p)$ in the case $p, q \in \mathbb{N}^n$ was completely solved in [9]. The case $p, q \in \mathbb{R}^n_{>0}$ was considered in [6], where the Authors characterized non-emptiness of $\operatorname{Prop}(\mathbb{E}_p, \mathbb{E}_q)$ and found $\operatorname{Aut}(\mathbb{E}_p)$. They did not give, however,

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the explicit form of an $F \in \mathrm{Prop}(\mathbb{E}_p, \mathbb{E}_q)$. The previous results are formulated in the following

Theorem 1 (cf. [9], [6]). *Assume that $n \geq 2$, $p, q \in \mathbb{R}^n_{>0}$.*

(a) The following conditions are equivalent

(i) $\mathrm{Prop}(\mathbb{E}_p, \mathbb{E}_q) \neq \varnothing$*;*

(ii) there exists $\sigma \in \Sigma_n$ such that $p_\sigma / q \in \mathbb{N}^n$.

(b) If $p, q \in \mathbb{N}^n$, then the following conditions are equivalent

(i) $F \in \mathrm{Prop}(\mathbb{E}_p, \mathbb{E}_q)$;

(ii) $F = \phi \circ \Psi_{p_\sigma/q} \circ \sigma$, where $\sigma \in \Sigma_n$ is such that $p_\sigma/q \in \mathbb{N}^n$ and $\phi \in \mathrm{Aut}(\mathbb{E}_q)$.

In particular, $\mathrm{Prop}(\mathbb{E}_p) = \mathrm{Aut}(\mathbb{E}_p)$*.*

(c) If $0 \leq k \leq n$, $p \in \{1\}^k \times (\mathbb{R}_{>0} \setminus \{1\})^{n-k}$, $z = (z', z_{k+1}, \dots, z_n)$, then the following conditions are equivalent

(i) $F = (F_1, \dots, F_n) \in \mathrm{Aut}(\mathbb{E}_p)$,

(ii) $F_j(z) := \begin{cases} H_j(z'), & \text{if } j \leq k \\ \zeta_j z_{\sigma(j)} \left(\frac{\sqrt{1-\|a'\|^2}}{1-\langle z', a' \rangle} \right)^{1/p_{\sigma(j)}}, & \text{if } j > k \end{cases}$*, where $\zeta_j \in \mathbb{T}$, $j > k$,*

$H = (H_1, \dots, H_k) \in \mathrm{Aut}(\mathbb{B}_k)$, $a' := H^{-1}(0)$, and $\sigma \in \Sigma_n(p)$.

In the general case thesis of Theorem 1 (b) is no longer true (take, for instance, $\Psi_{(2,2)} \circ H \circ \Psi_{(2,2)} \in \mathrm{Prop}(\mathbb{E}_{(2,2)}, \mathbb{E}_{(1/2,1/2)})$, where $H \in \mathrm{Aut}(\mathbb{B}_2)$, $H(0) \neq 0$).

Nevertheless, from the proof of Theorem 1.1 in [6] we easily derive the following theorem which will be of great importance during the investigation of proper holomorphic mappings between generalized Hartogs triangles.

Theorem 2. *Assume that $n \geq 2$, $p, q \in \mathbb{R}^n_{>0}$. Then the following conditions are equivalent*

(i) $F \in \mathrm{Prop}(\mathbb{E}_p, \mathbb{E}_q)$;

(ii) $F = \Psi_{p_\sigma/(qr)} \circ \phi \circ \Psi_r \circ \sigma$, where $\sigma \in \Sigma_n$ is such that $p_\sigma/q \in \mathbb{N}^n$, $r \in \mathbb{N}^n$ is such that $p_\sigma/(qr) \in \mathbb{N}^n$, and $\phi \in \mathrm{Aut}(\mathbb{E}_{p_\sigma/r})$.

In particular, $\mathrm{Prop}(\mathbb{E}_p) = \mathrm{Aut}(\mathbb{E}_p)$*.*

2. Generalized Hartogs triangles

Let $n, m \in \mathbb{N}$. For $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$ and $q = (q_1, \dots, q_m) \in \mathbb{R}^m_{>0}$, define the *generalized Hartogs triangle*

$$\mathbb{F}_{p,q} := \left\{ (z_1, \dots, z_n, w_1, \dots, w_m) \in \mathbb{C}^{n+m} : \sum_{j=1}^n |z_j|^{2p_j} < \sum_{j=1}^m |w_j|^{2q_j} < 1 \right\}.$$

Note that $\mathbb{F}_{p,q}$ is nonsmooth pseudoconvex Reinhardt domain, not containing the origin. Moreover, if $n = m = 1$, then $\mathbb{F}_{1,1}$ is the standard Hartogs triangle.

The problem of characterization of $\mathrm{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$ and $\mathrm{Aut}(\mathbb{F}_{p,q})$ has been investigated in many papers. The necessary and sufficient conditions for the nonemptiness of $\mathrm{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$ are given in [3] for $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}^m$, $n, m \geq 2$, in [4] for $p, \tilde{p} \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$, $n, m \geq 2$, and in [10] for $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}^m$, $m = 1$. The explicit form of an $F \in \mathrm{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$ is presented in [10] for $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}^m$, $m = 1$, whereas the description of $\mathrm{Aut}(\mathbb{F}_{p,q})$ may be found in [5] for $p \in \mathbb{N}^n$, $q \in \mathbb{N}^m$, $n, m \geq 2$, and in [10] for $p \in \mathbb{N}^n$, $q \in \mathbb{N}^m$, $m = 1$.

In the paper we shall only consider the case $n = 1$.

First we deal with the case $m = 1$.

Theorem 3. *Assume that $n = m = 1$, $p, q, \tilde{p}, \tilde{q} \in \mathbb{R}_{>0}$.*

(a) The following conditions are equivalent

(i) $\operatorname{Prop}(\mathbb{F}_{p,q},\mathbb{F}_{\tilde p,\tilde q})\neq\varnothing$;
(ii) *there exist* $k,l\in\mathbb{N}$ *such that* $l\tilde q/\tilde p-kq/p\in\mathbb{Z}$.

(b) *The following conditions are equivalent*
(i) $F\in\operatorname{Prop}(\mathbb{F}_{p,q},\mathbb{F}_{\tilde p,\tilde q})$;
(ii) $F(z,w)=\begin{cases}\left(\zeta z^k w^{l\tilde q/\tilde p-kq/p},\xi w^l\right), & \text{if } q/p\notin\mathbb{Q}\\ \left(\zeta z^{k'} w^{l\tilde q/\tilde p-k'q/p}B\left(z^{p'}w^{-q'}\right),\xi w^l\right), & \text{if } q/p\in\mathbb{Q}\end{cases}$,
where $\zeta,\xi\in\mathbb{T}$, $k,l\in\mathbb{N}$, $k'\in\mathbb{N}\cup\{0\}$ *are such that* $l\tilde q/\tilde p-kq/p\in\mathbb{Z}$, $l\tilde q/\tilde p-k'q/p\in\mathbb{Z}$, $p',q'\in\mathbb{Z}$ *are relatively prime with* $p/q=p'/q'$, *and* B *is a finite Blaschke product nonvanishing at 0 (it may happen that* $B\equiv 1$, *but then* $k'>0$*).*

In particular, $\operatorname{Prop}(\mathbb{F}_{p,q})\supsetneq\operatorname{Aut}(\mathbb{F}_{p,q})$.

(c) *The following conditions are equivalent*
(i) $F\in\operatorname{Aut}(\mathbb{F}_{p,q})$;
(ii) $F(z,w)=\left(w^{q/p}\phi(zw^{-q/p}),\xi w\right)$, *where* $\xi\in\mathbb{T}$, *and* $\phi\in\operatorname{Aut}(\mathbb{D})$ *with* $\phi(0)=0$ *whenever* $q/p\notin\mathbb{N}$.

Remark 4. The counterpart of the Theorem 3 for $p,q,\tilde p,\tilde q\in\mathbb{N}$ was proved (with minor mistakes) in [10], where it was claimed that $F\in\operatorname{Prop}(\mathbb{F}_{p,q},\mathbb{F}_{\tilde p,\tilde q})$ iff

$$F(z,w)=\begin{cases}\left(\zeta z^k w^{l\tilde q/\tilde p-kq/p},\xi w^l\right), & \text{if } q/p\notin\mathbb{N},\ l\tilde q/\tilde p-kq/p\in\mathbb{Z}\\ \left(\zeta w^{l\tilde q/\tilde p}B\left(zw^{-q/p}\right),\xi w^l\right), & \text{if } q/p\in\mathbb{N},\ l\tilde q/\tilde p\in\mathbb{N}\end{cases},\tag{1}$$

where $\zeta,\xi\in\mathbb{T}$, $k,l\in\mathbb{N}$, and B is a finite Blaschke product. Nevertheless, the mapping

$$\mathbb{F}_{2,3}\ni(z,w)\mapsto\left(z^3w^3B(z^2w^{-3}),w^3\right)\in\mathbb{F}_{2,5},$$

where B is nonconstant finite Blaschke product nonvanishing at 0, is proper holomorphic but not of the form (1). In fact, from the Theorem 3 (b) it follows immediately that for any choice of $p,q,\tilde p,\tilde q\in\mathbb{N}$ one may find mapping $F\in\operatorname{Prop}(\mathbb{F}_{p,q},\mathbb{F}_{\tilde p,\tilde q})$ having, as a factor of the first component, nonconstant Blaschke product nonvanishing at 0.

Our result gives a negative answer to the question posed by the Authors in [8], whether the structure of $\operatorname{Prop}(\mathbb{F}_{p,q},\mathbb{F}_{\tilde p,\tilde q})$ remains unchanged when passing from $p,q,\tilde p,\tilde q\in\mathbb{N}$ to arbitrary $p,q,\tilde p,\tilde q\in\mathbb{R}_{>0}$. It should be mentioned that, on the other hand, the automorphism group $\operatorname{Aut}(\mathbb{F}_{p,q})$ does not change when passing from $p,q\in\mathbb{N}$ to $p,q\in\mathbb{R}_{>0}$.

In the proof of Theorem 3, however, neither the method from [4] (where the assumption $m\geq 2$ is essential) nor the method from [10] (where the assumption $p,q,\tilde p,\tilde q\in\mathbb{N}$ is essential) can be used. Fortunately, it turns out that one may get Theorem 3 using part of the main result from [7], where complete characterization of nonelementary proper holomorphic mappings between bounded Reinhardt domains in $\mathbb{C}^2$ is given.

The remaining case $m\geq 2$ is considered in the following result.

Theorem 5. *Assume that* $n=1$, $m\geq 2$, $p,\tilde p\in\mathbb{R}_{>0}$, $q,\tilde q\in\mathbb{R}^m_{>0}$, $(z,w)\in\mathbb{C}\times\mathbb{C}^m$.

(a) *The following conditions are equivalent*
(i) $\operatorname{Prop}(\mathbb{F}_{p,q},\mathbb{F}_{\tilde p,\tilde q})\neq\varnothing$;
(ii) $p/\tilde p\in\mathbb{N}$ *and there exists* $\sigma\in\Sigma_m$ *such that* $q_\sigma/\tilde q\in\mathbb{N}^m$.

(b) *The following conditions are equivalent*
(i) $F\in\operatorname{Prop}(\mathbb{F}_{p,q},\mathbb{F}_{\tilde p,\tilde q})$;
(ii) $F(z,w)=(\zeta z^k,h(w))$, *where* $\zeta\in\mathbb{T}$, $k\in\mathbb{N}$, $h\in\operatorname{Prop}(\mathbb{E}_q,\mathbb{E}_{\tilde q})$, $h(0)=0$.

In particular, $\operatorname{Prop}(\mathbb{F}_{p,q})=\operatorname{Aut}(\mathbb{F}_{p,q})$.

(c) *The following conditions are equivalent*
(i) $F\in\operatorname{Aut}(\mathbb{F}_{p,q})$;

(ii) $F(z,w) = (\zeta z, h(w))$, *where* $\zeta \in \mathbb{T}$, $h \in \mathrm{Aut}(\mathbb{E}_q)$, $h(0) = 0$.

Theorem 5 (a) was proved in [3] (for $n, m \geq 2$, $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}^m$) and in [4] (for $n, m \geq 2$, $p, \tilde{p} \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$). Theorem 5 (b) was proved in [5] for $n, m \geq 2$, $p = \tilde{p} \in \mathbb{N}^n$, $q = \tilde{q} \in \mathbb{N}^m$. Theorem 5 (c) was proved in [5] for $n, m \geq 2$, $p \in \mathbb{N}^n$, $q \in \mathbb{N}^m$. Part (c) of Theorem 5 gives an affirmative answer to the question posed by the Authors in [8], whether the description of the automorphism group (c) remains true for arbitrary $p \in \mathbb{R}^n_{>0}$, $q \in \mathbb{R}^m_{>0}$ (at least in the case $n = 1$).

Remark 6. Using Barrett's and Bell's results (cf. [1], [2]) one may show (cf. [3]) that any $F \in \mathrm{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$ extends holomorphically past any boundary point $(z,w) \in \partial\mathbb{F}_{p,q} \setminus \{(0,0)\}$.

Let

$$K = K_{p,q} := \left\{ (z,w) \in \mathbb{C}^{n+m} : 0 < \sum_{j=1}^{n} |z_j|^{2p_j} = \sum_{j=1}^{m} |w_j|^{2q_j} < 1 \right\},$$

$$L = L_{p,q} := \left\{ (z,w) \in \mathbb{C}^{n+m} : \sum_{j=1}^{n} |z_j|^{2p_j} < \sum_{j=1}^{m} |w_j|^{2q_j} = 1 \right\}.$$

Analogically we define $\tilde{K} := K_{\tilde{p},\tilde{q}}$ and $\tilde{L} := L_{\tilde{p},\tilde{q}}$.

For $m > 1$ it is shown in [4] that

$$F(K) \subset \tilde{K}, \quad F(L) \subset \tilde{L}.$$

for any $F \in \mathrm{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$.

3. Proofs

Proof of Theorem 2. The implication (ii)⇒(i) is obvious.

To prove (i)⇒(ii) let $F = (F_1, \dots, F_n) \in \mathrm{Prop}(\mathbb{E}_p, \mathbb{E}_q)$.

Following [11] any automorphism $H = (H_1, \dots, H_n) \in \mathrm{Aut}(\mathbb{B}_n)$ is of the form

$$H_j(z) = \frac{\sqrt{1 - \|a\|^2}}{1 - \langle z, a \rangle} \sum_{k=1}^{n} h_{j,k}(z_k - a_k), \quad z = (z_1, \dots, z_n) \in \mathbb{B}_n,\ j = 1, \dots, n,$$

where $a = (a_1, \dots, a_n) \in \mathbb{B}_n$ and $Q = [h_{j,k}]$ is an $n \times n$ matrix such that

$$\bar{Q}(\mathbb{I}_n - \bar{a}\,{}^t a)\,{}^t Q = \mathbb{I}_n,$$

where $\mathbb{I}_n$ is the unit $n \times n$ matrix, whereas $\bar{A}$ (resp. ${}^tA$) is the conjugate (resp. transpose) of an arbitrary matrix A. In particular, Q is unitary if $a = 0$.

It follows from [6] that there exists $\sigma \in \Sigma_n$ such that $p_\sigma / q \in \mathbb{N}^n$, $h_{j,\sigma(j)} \neq 0$, and

$$F_j(z) = \left(\frac{\sqrt{1 - \|a\|^2}}{1 - \langle z^p, a \rangle} h_{j,\sigma(j)} z_{\sigma(j)}^{p_{\sigma(j)}} \right)^{1/q_j} \tag{2}$$

whenever $1/q_j \notin \mathbb{N}$.

If $1/q_j \in \mathbb{N}$ then F_j either is of the form (2), where $p_{\sigma(j)}/q_j \in \mathbb{N}$, or

$$F_j(z) = \left(\frac{\sqrt{1 - \|a\|^2}}{1 - \langle z^p, a \rangle} \sum_{k=1}^{n} h_{j,k}(z_k^{p_k} - a_k) \right)^{1/q_j}$$

where $p_k \in \mathbb{N}$ for any k such that $h_{j,k} \neq 0$.

Consequently, if we define $r = (r_1, \dots, r_n)$ as

$$r_j := \begin{cases} p_{\sigma(j)}, & \text{if } a_{\sigma(j)} \neq 0 \text{ or there is } k \neq \sigma(j) \text{ with } h_{j,k} \neq 0 \\ p_{\sigma(j)}/q_j, & \text{otherwise} \end{cases},$$

then it is easy to see that $r \in \mathbb{N}^n$, $p_\sigma/(qr) \in \mathbb{N}^n$, and F is as desired. $\square$

Remark 7. Note that in the case $p, q \in \mathbb{N}^n$ we have $1/q_j \in \mathbb{N}$ iff $q_j = 1$. Hence the above definition of r implies that $r = p_\sigma/q$ and, consequently, we get thesis of Theorem 1 (b).

Proof of Theorem 3. Observe, that (a) and (c) follows immediately from (b). Thus, it suffices to prove part (b).

The implication (ii)$\Rightarrow$(i) in part (b) holds for any $p, q, \tilde{p}, \tilde{q} > 0$. Indeed, if $F = (G, H)$ is of the form given in (ii), then

$$|G(z,w)|^{\tilde{p}}|H(z,w)|^{-\tilde{q}} = \begin{cases} \left(|z||w|^{-q/p}\right)^{k\tilde{p}}, & \text{if } q/p \notin \mathbb{Q} \\ \left(|z||w|^{-q/p}\right)^{k'\tilde{p}} \left|B(z^{p'}w^{-q'})\right|^{\tilde{p}}, & \text{if } q/p \in \mathbb{Q} \end{cases}.$$

To prove the implication (i)$\Rightarrow$(ii) in (b), let $F \in \operatorname{Prop}(\mathbb{F}_{p,q}, \mathbb{F}_{\tilde{p},\tilde{q}})$.

Assume first that F is elementary algebraic mapping, i.e. it is of the form

$$F(z,w) = \left(\alpha z^a w^b, \beta z^c w^d\right),$$

where $a, b, c, d \in \mathbb{Z}$ are such that $ad - bc \neq 0$ and $\alpha, \beta \in \mathbb{C}$ are some constants. Since F is surjective, we infer that $c = 0$, $d \in \mathbb{N}$, and $\xi := \beta \in \mathbb{T}$. Moreover,

$$|\alpha|^{\tilde{p}}|z|^{a\tilde{p}}|w|^{b\tilde{p}-d\tilde{q}} < 1, \tag{3}$$

whence $a \in \mathbb{N}$, $b\tilde{p} - d\tilde{q} \in \mathbb{N}$, and $\zeta := \alpha \in \mathbb{T}$. Let $k := a$, $l := d$. One may rewrite (3) as

$$\left(|z|^p|w|^{-q}\right)^{k\tilde{p}/p}|w|^{b\tilde{p}-l\tilde{q}+kq\tilde{p}/p} < 1.$$

Since one may take sequence $(z_\nu, 1/2)_{\nu\in\mathbb{N}} \subset \mathbb{F}_{p,q}$ with $|z_\nu|^p 2^q \to 1$ as $\nu \to \infty$, we infer that $b\tilde{p} - l\tilde{q} + kq\tilde{p}/p = 0$, i.e.

$$b = \frac{l\tilde{q}}{\tilde{p}} - \frac{kq}{p}.$$

Consequently, F is as in the condition (ii) of the Theorem 3 (b).

Assume now that F is not elementary. Then from the Theorem 0.1 in [7] it follows that F is of the form

$$F(z,w) = \left(\alpha z^a w^b \tilde{B}\left(z^{p'}w^{-q'}\right), \beta w^l\right),$$

where $a, b \in \mathbb{Z}$, $a \geq 0$, $p', q', l \in \mathbb{N}$, p', q' are relatively prime,

$$\frac{q'}{p'} = \frac{q}{p}, \quad \frac{\tilde{q}}{\tilde{p}} = \frac{aq' + bp'}{lp'}, \tag{4}$$

$\alpha, \beta \in \mathbb{C}$ are some constants, and $\tilde{B}$ is a nonconstant finite Blaschke product nonvanishing at the origin.

From the surjectivity of F we immediately infer that $\zeta := \alpha \in \mathbb{T}$ and $\xi := \beta \in \mathbb{T}$. If we put $k' := a$, then (4) implies

$$b = \frac{l\tilde{q}}{\tilde{p}} - \frac{k'q}{p},$$

which ends the proof. □

Proof of Theorem 5. We shall write $w = (w_1, \ldots, w_m) \in \mathbb{C}^m$. We may assume without loss of generality that there is $0 \leq \mu \leq m$ with $\tilde{q} \in \{1\}^\mu \times (\mathbb{R}_{>0} \setminus \{1\})^{m-\mu}$. Let

$$F = (G, H) : \mathbb{F}_{p,q} \to \mathbb{F}_{\tilde{p},\tilde{q}} \subset \mathbb{C} \times \mathbb{C}^m$$

be proper holomorphic mapping. It follows from [4] that $F(L) \subset \tilde{L}$ and H is independent of the variable z. Hence $h := H(0, \cdot) \in \operatorname{Prop}((\mathbb{E}_q)_*, (\mathbb{E}_{\tilde{q}})_*)$. Consequently, by Hartogs theorem $h \in \operatorname{Prop}(\mathbb{E}_q, \mathbb{E}_{\tilde{q}})$, i.e. (Theorem 2)

$$h = \Psi_{q_\sigma/(\tilde{q}r)} \circ \psi \circ \Psi_r \circ \sigma$$

for some $\sigma \in \Sigma_m$ with $q_\sigma/\tilde{q} \in \mathbb{N}^m$, $r \in \mathbb{N}^m$ with $q_\sigma/(\tilde{q}r) \in \mathbb{N}^m$, and $\psi \in \operatorname{Aut}(\mathbb{E}_{q_\sigma/r})$ with $\psi(0) = 0$. Indeed, if $a = (a_1, \dots, a_m)$ is a zero of h we immediately get

$$G(z, a) = 0, \quad |z|^{2p} < \sum_{j=1}^m |a_j|^{2q_j},$$

which is clearly a contradiction, unless $a = 0$.

Without loss of generality we may assume that there is $\mu \le l \le m$ with $1/\tilde{q}_j \notin \mathbb{N}$ iff $j > l$. It follows from the proof of Theorem 2 that

$$\frac{q_{\sigma(j)}}{r_j} = \begin{cases} 1, & \text{if } j = 1, \dots, l \\ \tilde{q}_j, & \text{if } j = l+1, \dots, m \end{cases},$$

whence

$$\psi(w) = (U(w_1, \dots, w_l), \xi_{l+1} w_{l+\tau(1)}, \dots, \xi_m z_{l+\tau(m-l)}),$$

where $U = (U_1, \dots, U_l) \in \mathbb{U}(l)$ and $\tau \in \Sigma_{m-l}(\tilde{q}_{l+1}, \dots, \tilde{q}_m)$. Finally,

$$h(w) = \Big(U_1^{1/\tilde{q}_1}\left(w_{\sigma(1)}^{q_{\sigma(1)}}, \dots, w_{\sigma(l)}^{q_{\sigma(l)}}\right), \dots, U_l^{1/\tilde{q}_l}\left(w_{\sigma(1)}^{q_{\sigma(1)}}, \dots, w_{\sigma(l)}^{q_{\sigma(l)}}\right), \\ \xi_{l+1} w_{\sigma(l+1)}^{q_{\sigma(l+1)}/\tilde{q}_{l+1}}, \dots, \xi_m w_{\sigma(m)}^{q_{\sigma(m)}/\tilde{q}_m} \Big).$$

In particular, if we write $h = (h_1, \dots, h_m)$,

$$\sum_{j=1}^m |h_j(w)|^{2\tilde{q}_j} = \sum_{j=1}^m |w_j|^{2q_j}, \quad w \in \mathbb{E}_q. \tag{5}$$

For $w \in \mathbb{C}^m$, $0 < \rho_w := \sum_{j=1}^m |w_j|^{2q_j} < 1$ let

$$g(z) := G(z, w), \quad z \in \rho_w^{1/(2p)}\mathbb{D}.$$

g may depend, a priori, on w. Since $F(K) \subset \tilde{K}$, it follows from (5) that $g \in \operatorname{Prop}\left(\rho_w^{1/(2p)}\mathbb{D}, \rho_w^{1/(2\tilde{p})}\mathbb{D}\right)$, i.e.

$$g(z) = \rho_w^{1/(2\tilde{p})} B\left(z \rho_w^{-1/(2p)}\right), \quad z \in \rho_w^{1/(2p)}\mathbb{D}, \tag{6}$$

where B is a finite Blaschke product. Let

$$\mathbb{F}^0_{p,q} := \mathbb{F}_{p,q} \cap \left(\mathbb{C} \times \{0\}^{\sigma(1)-1} \times \mathbb{C} \times \{0\}^{m-\sigma(1)}\right),$$
$$\mathbb{F}^0_{\tilde{p},q_\sigma/r} := \mathbb{F}_{\tilde{p},q_\sigma/r} \cap \left(\mathbb{C}^2 \times \{0\}^{m-1}\right).$$

Let $\Phi \in \operatorname{Aut}(\mathbb{F}_{\tilde{p},q_\sigma/r})$ be defined by

$$\Phi(z, w) := \left(z, U^{-1}(w_1, \dots, w_l), w_{l+1}, \dots, w_m\right)$$

and let

$$\hat{\xi}_1 := \begin{cases} \xi_1, & \text{if } l = 0 \\ 1, & \text{if } l > 0 \end{cases}, \qquad \hat{q}_1 := \begin{cases} \tilde{q}_1, & \text{if } l = 0 \\ 1, & \text{if } l > 0 \end{cases}.$$

Then $\Phi \circ (G, \psi \circ \Psi_r \circ \sigma) \in \operatorname{Prop}(\mathbb{F}^0_{p,q}, \mathbb{F}^0_{\tilde{p},q_\sigma/r})$ with

$$(\Phi \circ (G, \psi \circ \Psi_r \circ \sigma))(z, w) = \left(G(z, w), \hat{\xi}_1 w_{\sigma(1)}^{q_{\sigma(1)}/\hat{q}_1}, 0, \dots, 0\right), \quad (z, w) \in \mathbb{F}^0_{p,q}. \tag{7}$$

It follows from Theorem 3 that

$$(\Phi \circ (G, \psi \circ \Psi_r \circ \sigma))(z, w) = \left(\hat{G}(z, w), \eta w_{\sigma(1)}^s, 0, \dots, 0\right), \quad (z, w) \in \mathbb{F}^0_{p,q}, \tag{8}$$

where

$$\hat{G}(z, w) := \begin{cases} \zeta z^k w_{\sigma(1)}^{s\hat{q}_1/\tilde{p} - kq_{\sigma(1)}/p}, & \text{if } q_{\sigma(1)}/p \notin \mathbb{Q} \\ \zeta z^{k'} w_{\sigma(1)}^{s\hat{q}_1/\tilde{p} - k'q_{\sigma(1)}/p} \hat{B}\left(z^{p'} w_{\sigma(1)}^{-q'_{\sigma(1)}}\right), & \text{if } q_{\sigma(1)}/p \in \mathbb{Q} \end{cases},$$

$\zeta, \eta \in \mathbb{T}$, $k, s, p', q'_{\sigma(1)} \in \mathbb{N}$, $k' \in \mathbb{N} \cup \{0\}$ are such that $p', q'_{\sigma(1)}$ are relatively prime, $q_{\sigma(1)}/p = q'_{\sigma(1)}/p'$, $s\hat{q}_1/\tilde{p} - kq_{\sigma(1)}/p \in \mathbb{Z}$, $q_{\sigma(1)}/p = q'_{\sigma(1)}/p'$, $s\hat{q}_1/\tilde{p} - kq_{\sigma(1)}/p \in \mathbb{Z}$, and $\hat{B}$ is a finite Blaschke product nonvanishing at 0 (if $\hat{B} \equiv 1$ then $k' > 0$). Hence

$$(9)\quad (\Phi \circ (G, \psi \circ \Psi_r \circ \sigma))(z, w) = \\ \left(\hat{G}(z, w) + \alpha(z, w), w_{\sigma(1)}^{q_{\sigma(1)}}, \ldots, w_{\sigma(l)}^{q_{\sigma(l)}}, \xi_{l+1} w_{\sigma(l+1)}^{q_{\sigma(l+1)}/\tilde{q}_{l+1}}, \ldots, \xi_m w_{\sigma(m)}^{q_{\sigma(m)}/\tilde{q}_m}\right),$$

for $(z, w) \in \mathbb{F}_{p,q}$, $w_{\sigma(1)} \neq 0$, where α is holomorphic on $\mathbb{F}_{p,q}$ with $\alpha|_{\mathbb{F}^0_{p,q}} = 0$. Comparing (7) and (8) we conclude that

$$\eta = \hat{\xi}_1, \quad s = q_{\sigma(1)}/\hat{q}_1.$$

Since the mapping on the left side of (9) is holomorphic on $\mathbb{F}_{p,q}$, the function

$$(10)\qquad \hat{G}(z, w) = \begin{cases} \zeta z^k w_{\sigma(1)}^{q_{\sigma(1)}(1/\tilde{p} - k/p)}, & \text{if } q_{\sigma(1)}/p \notin \mathbb{Q} \\ \zeta z^{k'} w_{\sigma(1)}^{q_{\sigma(1)}(1/\tilde{p} - k'/p)} \hat{B}\left(z^{p'} w_{\sigma(1)}^{-q'_{\sigma(1)}}\right), & \text{if } q_{\sigma(1)}/p \in \mathbb{Q} \end{cases}$$

with $q_{\sigma(1)}(1/\tilde{p} - k/p) \in \mathbb{Z}$ and $q_{\sigma(1)}(1/\tilde{p} - k'/p) \in \mathbb{Z}$ has to be holomorphic on $\mathbb{F}_{p,q}$, too. Since $m \geq 2$, it may happen $w_{\sigma(1)} = 0$. Consequently, $q_{\sigma(1)}(1/\tilde{p} - k/p) \in \mathbb{N} \cup \{0\}$ in the first case of (10), whereas $\hat{B}(t) = t^{k''}$ for some $k'' \in \mathbb{N}$ with $q_{\sigma(1)}(1/\tilde{p} - k'/p) - k'' q'_{\sigma(1)} \in \mathbb{N} \cup \{0\}$ in the second case. Thus

$$\hat{G}(z, w) = \zeta z^k w_{\sigma(1)}^{q_{\sigma(1)}(1/\tilde{p} - k/p)},$$

where $k \in \mathbb{N}$, $q_{\sigma(1)}(1/\tilde{p} - k/p) \in \mathbb{N} \cup \{0\}$ (in the second case of (10) it suffices to take $k := k' + p'k''$).

Observe that $\hat{G} + \alpha = G$. Fix $w \in \{0\}^{\sigma(1)-1} \times \mathbb{C} \times \{0\}^{m-\sigma(1)}$ with $0 < \rho_w < 1$. Then $\rho_w = |w_{\sigma(1)}|^{2q_{\sigma(1)}}$ and $\hat{G}(\cdot, w) = g$ on $\rho_w^{1/(2p)}\mathbb{D}$, i.e.

$$\zeta z^k w_{\sigma(1)}^{q_{\sigma(1)}(1/\tilde{p} - k/p)} = |w_{\sigma(1)}|^{q_{\sigma(1)}/\tilde{p}} B\left(z |w_{\sigma(1)}|^{-q_{\sigma(1)}/p}\right), \quad z \in |w_{\sigma(1)}|^{q_{\sigma(1)}/p}\mathbb{D}.$$

Hence $B(t) = \zeta t^k$ and $q_{\sigma(1)}(1/\tilde{p} - k/p) = 0$, i.e. $k = p/\tilde{p}$. Hence part (a) is proved. To finish part (b), note that $g(z) = \zeta z^{p/\tilde{p}}$. Consequently, g does not depend on w and

$$G(z, w) = \zeta z^{p/\tilde{p}}, \quad (z, w) \in \mathbb{F}_{p,q}.$$

Part (c) follows directly from (b). □

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